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Perturbations of Volterra Integral Equations

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1. INTRODUCTION

Consider the system of Volterra integral equations

$$x(t) = f(t) - \int_0^t a(t, s)\{x(s) + g(s, x(s))\}\, ds \qquad (t \geqslant 0), \tag{1.1}$$

where x, f and g are vectors in E_n, a is an n by n matrix, and E_n is n-dimensional real or complex Euclidean space. Let $g(t, x) = o(|x|)$ as $|x| \to 0$ uniformly for $0 \leqslant t < \infty$, where $|\ |$ denotes any vector norm in E_n. We shall compare in various ways the solution of (1.1) with the solution of the (unperturbed) linear system

$$y(t) = f(t) - \int_0^t a(t, s)\, y(s)\, ds \qquad (t \geqslant 0). \tag{1.2}$$

In Theorems 1-3 we present, roughly speaking, sufficient conditions in order that a certain stability of the unperturbed equation (1.2) implies a corresponding local stability of Eq. (1.1). In Theorems 4 and 5 we apply the perturbation method of the earlier results to establish sufficient conditions for the existence of asymptotically periodic and almost periodic solutions of (1.1). Finally in Theorem 6 we illustrate the theory by means of a general example which is not of convolution type. Our methods rest on the use of the resolvent kernel $r(t, s)$ for the linear system (1.2) and the familiar contraction principle. Our assumptions on the resolvent kernel may be viewed as sufficient condi-

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tions in order to insure the admissibility of various spaces under the integral operator

$$K\varphi(t) = \int_0^t r(t, s)\, \varphi(s)\, ds \qquad (t \geqslant 0).$$

Linearization of Volterra equations with the kernel of convolution type, $a(t, s) = a(t - s)$, has recently been studied in [1] and [2]. These results deal with the local stability problem for (1.1) but not with asymptotically periodic or almost periodic solutions. Here we generalize considerably the results obtained earlier and we simplify some of the arguments.

Corduneanu [3], [4] has studied perturbation problems for integral equations by adapting the admissibility theory of Massera and Schaffer [5]. Antosiewicz [6], [7] has obtained some general results which may be used to study the asymptotic behavior of solutions of Volterra equations. However, these results are not strictly comparable to ours since neither author considers the problem of comparing solutions of (1.1) and (1.2) when these equations do not reduce to ordinary differential equations. Levin and Nohel [8], Levin [9], and Corduneanu [10] have studied perturbation problems for certain Volterra integro-differential equations. They obtain results of a different nature by means of energy methods.

2. Summary of Results

a. *Perturbation Theory*

The resolvent system corresponding to the system (1.2) is

$$r(t, s) = a(t, s) - \int_s^t a(t, u)\, r(u, s)\, du \qquad (0 \leqslant s \leqslant t < \infty) \tag{2.1}$$

and its solution $r(t, s)$ is called the resolvent kernel. If $a(t, s)$ is locally L^1 in (t, s) and if $r(t, s)$ exists and is locally L^1 in (t, s), then system (1.1) may be written in the equivalent form ("variation of constants formula")

$$x(t) = y(t) - \int_0^t r(t, s)\, g(s, x(s))\, ds, \tag{2.2}$$

where $y(t)$ is the solution of the linear system (1.2) given by

$$y(t) = f(t) - \int_0^t r(t, s) f(s)\, ds. \tag{2.3}$$

It is always assumed that f is locally integrable and that, unless otherwise stated, all equations are satisfied in an almost everywhere sense.

Throughout the paper we shall assume that the following hypotheses are satisfied:

(H1) $r(t, s)$ exists, is locally integrable in (t, s) for $0 \leqslant s \leqslant t < \infty$ and satisfies (2.1).

(H2) $g(t, x)$ is measurable in (t, x) for $0 \leqslant t < \infty$ and $|x| < \infty$ and $g(t, 0) \equiv 0$.

(H3) For each $\mathscr{E} > 0$, there exists $\eta > 0$ such that

$$| g(t, x) - g(t, y) | \leqslant \mathscr{E} | x - y | ,$$

uniformly in $t \geqslant 0$ whenever $|x|, |y| \leqslant \eta$.

Additional hypotheses will be stated as they are needed.

For any function $\varphi \in L^\infty[0, \infty)$, let $\| \varphi \|_\infty$ be the norm of φ considered as an element of $L^\infty[0, \infty)$. Similarly let $\| \, \|_p$ denote the norm of elements of $L^p[0, \infty)$.

THEOREM 1. *Suppose that the solution $y(t)$ of the linear system* (1.2) *satisfies $y \in L^\infty[0, \infty)$ and that there exists a constant $B > 0$ such that*

$$\int_0^t | r(t, s) | \, ds \leqslant B \qquad (t \geqslant 0). \tag{2.4}$$

For any fixed $\lambda \in (0, 1)$ there exists a number $\mathscr{E}_0 > 0$ such that for $0 < \mathscr{E} \leqslant \mathscr{E}_0$, if $\| y \|_\infty \leqslant \lambda \mathscr{E}$, then there exists a unique solution $x(t)$ of the system (1.1), *$x \in L^\infty[0, \infty)$ and in addition $\| x \|_\infty \leqslant \mathscr{E}$.*

Uniqueness here means uniqueness within the class $L^\infty[0, \infty)$ of solutions for which $\| x \|_\infty \leqslant \mathscr{E}$. Note that if $f \in L^\infty[0, \infty)$, then assumption (2.4) used in (2.3) implies that $y \in L^\infty[0, \infty)$. Thus Theorem 1 may be regarded as a stability result for the system (1.1) in the following sense. For every $\mathscr{E} > 0$ and sufficiently small, there exists a $\delta > 0$ such that for every $f \in L^\infty[0, \infty)$ with $\| f \|_\infty < \delta$ the solution x of (1.1) is in $L^\infty[0, \infty)$ and $\| x \|_\infty \leqslant \mathscr{E}$.

We now consider the case where the solution $y(t)$ of the linear system (1.2) is also absolutely integrable.

THEOREM 2. *Let* (2.4) *be satisfied. Suppose that $y \in L^\infty[0, \infty) \cap L^1(0, \infty)$ and that there exists a constant $A > 0$ such that*

$$\sup \left\{ \int_s^T | r(t, s) | \, dt : 0 \leqslant s \leqslant T < \infty \right\} = A < \infty. \tag{2.5}$$

There exists a number $\mathscr{E}_0 > 0$ such that for $0 < \mathscr{E} \leqslant \mathscr{E}_0$, if $\| y \|_\infty \leqslant \lambda \mathscr{E}$ for some fixed $\lambda \in (0, 1)$, then there exists a unique solution $x(t)$ of the system (1.1), *$x \in L^\infty[0, \infty) \cap L^1(0, \infty)$ and in addition $\| x \|_\infty \leqslant \mathscr{E}$.*

The remarks following Theorem 1 concerning uniqueness and stability apply here as well with $f \in L^1[0, \infty)$ and (2.5) used in addition to (2.4) in order that $y \in L^1(0, \infty)$.

If $r(t, s) = r(t - s)$, then the assumption $r \in L^1(0, \infty)$ implies both assumptions (2.4) and (2.5). In this case Theorem 2 may be improved as follows.

THEOREM 2′. *Let $r(t, s) = r(t - s)$ and let $r \in L^1(0, \infty)$. Let $p \geqslant 1$ and let $y \in L^\infty[0, \infty) \cap L^p(0, \infty)$. There exists a number $\mathscr{E}_0 > 0$ such that for $0 < \mathscr{E} \leqslant \mathscr{E}_0$ if $\| y \|_\infty \leqslant \lambda\mathscr{E}$ for some fixed $\lambda \in (0, 1)$, then there exists a unique solution $x(t)$ of the system (1.1), $x \in L^\infty[0, \infty) \cap L^p(0, \infty)$ and in addition $\| x \|_\infty \leqslant \mathscr{E}$.*

If in Theorem 2′ it is assumed that $\| y \|_p$ is small, then one also has that $\| x \|_p$ is small. Note that if $\| f \|_p$ is small, it follows from (2.3) that $\| y \|_p$ will also be small.

We next consider the case when $y(t)$ is a bounded continuous function from $[0, \infty)$ into E_n. Let $BC[0, \infty)$ denote the set of all such functions and if $\varphi \in BC[0, \infty)$ define its norm by $\| \varphi \| = \sup\{| \varphi(t) | : 0 \leqslant t < \infty\}$.

THEOREM 3. *Let $y \in BC[0, \infty)$, let $r(t, s)$ satisfy (2.4) and let*

$$\lim_{h \to 0} \left(\int_t^{t+h} | r(t + h, s) | \, ds + \int_0^t | r(t + h, s) - r(t, s) | \, ds \right) = 0, \tag{2.6}$$

for each $t \geqslant 0$. There exists a number $\mathscr{E}_0 > 0$ such that for $0 < \mathscr{E} \leqslant \mathscr{E}_0$, if $\| y \| \leqslant \lambda\mathscr{E}$ for some fixed $\lambda \in (0, 1)$, then there exists a unique solution $x(t)$ of the system (1.1), $x \in BC[0, \infty)$ and in addition $\| x \| \leqslant \mathscr{E}$.

The remarks made previously concerning uniqueness and stability are again applicable with $\| \, \|_\infty$ replaced by $\| \, \|$. Moreover the solution $x(t)$ satisfies (1.1) everywhere. Note that if $r(t, s) = r(t - s)$ and if $r \in L^1(0, \infty)$, then assumption (2.6) is automatically satisfied.

As a consequence of Theorem 3 we have the following result.

COROLLARY 3.1. *If the hypotheses of Theorem 3 are satisfied, if for each $T > 0$*

$$\lim_{t \to \infty} \int_0^T | r(t, s) | \, ds = 0, \tag{2.7}$$

and if $\lim y(t) = 0$, $t \to \infty$, *then* $x(t) \to 0$ *as* $t \to \infty$.

If $r(t, s) = r(t - s)$ and $r \in L^1(0, \infty)$, then (2.7) is again automatically satisfied. Moreover, if $f \in BC[0, \infty)$ and $f(t) \to 0$ as $t \to \infty$, then it follows from (2.3) that $y(t) \to 0$. Thus Corollary 3.1 is a type of asymptotic stability theorem for the system (1.1).

b. *Asymptotically Periodic and Almost Periodic Solutions*

A function $\varphi \in BC[0, \infty)$ is called asymptotically *ω-periodic* if and only if there exists a function $p \in BC(-\infty, \infty)$ where p is ω-periodic, such that $\lim(\varphi(t) - p(t)) = 0$, as $t \to \infty$. In this case we write $\varphi \sim p$. The existence of asymptotically ω-periodic solutions of (1.1) is established in the following result.

THEOREM 4. *Let $y(t)$ be asymptotically ω-periodic and let $r(t, s)$ satisfy* (2.4), (2.6), (2.7) *as well as*

$$r(t + \omega, s + \omega) = r(t, s) \qquad (-\infty < s \leqslant t < \infty), \tag{2.8}$$

and uniformly in $t \geqslant 0$

$$\lim_{\substack{n,m\to\infty \\ n>m}} \int_0^{(n-m)\omega} | r(t + n\omega, s) | \, ds = 0. \tag{2.9}$$

Let $g(t, x)$ be ω-periodic in t for each x. There exists a number $\mathscr{E}_0 > 0$ such that for $0 < \mathscr{E} \leqslant \mathscr{E}_0$, if $\| y \| \leqslant \lambda\mathscr{E}$ for some fixed $\lambda \in (0, 1)$, then there exists a unique solution $x(t)$ of the system (1.1) *which is asymptotically ω-periodic and in addition $\| x \| \leqslant \mathscr{E}$.*

A sufficient condition for the solution $y(t)$ of the linear system (1.2) to be asymptotically ω-periodic is given by the following result. This result shows why in general one cannot expect periodic solutions.

LEMMA 4.1. *Let $f \in BC[0, \infty)$, let f be ω-periodic, let $r(t, s)$ satisfy* (2.4), (2.6), (2.7), (2.8) *and* (2.9). *Then the solution $y(t)$ of* (1.2) *given by* (2.3) *is asymptotically ω-periodic.*

If $r(t, s)$ satisfies the condition

$$\lim_{t\to+\infty} \int_{-\infty}^{0} | r(t, s) | \, ds = 0,$$

then $r(t, s)$ satisfies (2.9) for any $\omega > 0$. If the kernel $a(t, s)$ in (1.1) is of convolution type, then the resolvent $r(t, s) = r(t - s)$ is of convolution type and $r(t + \omega, s + \omega) = r(t, s) = r(t - s)$ for all $\omega > 0$. If in addition $r(t) \in L^1(0, \infty)$ one readily sees that the hypotheses (2.4)-(2.9) are satisfied. Thus Lemma 4.1 and Theorem 4 include convolution equations as special cases. For the latter we may also consider asymptotically constant solutions.

A deep of this type was obtained by Levin [12]. Here we establish a different result.

COROLLARY 4.2. *Let* $r(t, s) = r(t - s)$, $r \in L^1(0, \infty)$, $g(t, x) \equiv g(x)$ *independent of* t, *and* $\lim y(t) = y(\infty)$ *as* $t \to \infty$. *If* $\lambda \in (0, 1)$ *is fixed, then there exists* $\mathscr{E}_0 > 0$ *such that when* $0 < \mathscr{E} \leqslant \mathscr{E}_0$ *and* $\| y \| \leqslant \lambda\mathscr{E}$, *the unique solution* $x(t)$ *of the system* (1.1) *exists for* $0 \leqslant t < \infty$, $\lim_{t\to\infty} x(t) = x(\infty)$ *exists and this limit satisfies the equation*

$$x(\infty) = y(\infty) + \left(\int_0^\infty r(s)\, ds\right) g(x(\infty)).$$

In order to study the existence of asymptotically almost periodic solutions of (1.1) we need to recall some standard definitions [13]. Let $AP(E_n)$ be the set of all continuous functions $G(t, x)$ such that G is almost periodic in t uniformly for x on compact subsets of E_n. Define semi-norms on $AP(E_n)$ by

$$d_m(G) = \sup\{| G(t, x) |; -\infty < t < \infty, | x | \leqslant m\}$$

and define a metric by

$$d(G_1, G_2) = \sum_{m=1}^{\infty} \frac{2^{-m} d_m(G_1 - G_2)}{1 + d_m(G_1 - G_2)}.$$

If $G \in AP(E_n)$, the module of G in $E_m(\mathrm{Mod}_m(G))$ is the set of all almost periodic functions $p : R^1 \to E_m$ with the property that if $\{t_n\}$ is any real sequence such that the translates $\{G(t + t_n)\}$ form a Cauchy sequence in the metric d, then the translates $\{p(t + t_n)\}$ form a Cauchy sequence in the sup norm $\| \; \|$. It is readily shown that $\mathrm{Mod}_m(G)$ is a closed subspace of the set of all almost periodic functions $p : R^1 \to E_m$ with respect to the sup norm.

DEFINITION. *A function* $\varphi \in BC[0, \infty)$ *is called asymptotically almost periodic* $(\mathrm{Mod}_n(G))$ *if and only if there exists a function* $p \in \mathrm{Mod}_n(G)$ *such that* $\varphi(t) - p(t) \to 0$ *as* $t \to \infty$. *In this case we shall write* $\varphi \approx p$.

Theorem 4 has an analogue for the existence of asymptotically almost periodic solutions. For simplicity we confine ourselves to the convolution case of (1.1), $a(t, s) = a(t - s)$.

THEOREM 5. *Let* $r(t, s) = r(t - s)$, $r \in L^1(0, \infty)$. *Suppose that* f *is continuous and almost periodic and that* $g(t, x) \in AP(E_n)$. *Then for any fixed* $\lambda \in (0, 1)$ *there exists* $\mathscr{E}_0 > 0$ *such that if* $0 < \mathscr{E} \leqslant \mathscr{E}_0$ *and* $\| y \| \leqslant \lambda\mathscr{E}$, *then there exists a unique solution* $x(t)$ *of system* (1.1) *which is asymptotically almost periodic* $\mathrm{Mod}_n(f, g)$ *and satisfies* $\| x \| \leqslant \mathscr{E}$.

In the above statement (f, g) is regarded as an element of $AP(E_{2n})$.

c. *Example*

We consider an example of the kernel $a(t, s)$ of nonconvolution type which gives rise to a resolvent kernel $r(t, s)$ satisfying some or all of the hypotheses of the theorems stated above. As we have already pointed out if $r(t, s) = r(t - s)$ and $r \in L^1(0, \infty)$, then all assumptions previously imposed are satisfied. Let $n = 1$ and let

$$a(t, s) = A(t - s)\, B(s) \tag{2.10}$$

where A and B satisfy the following assumptions:

(A1) $A(t)$ is positive, continuous and locally Lebesque integrable on the interval $0 < t < \infty$;

(A2) $A(t)$ is nonincreasing on $0 < t < \infty$ and for each $T > 0$ the function $A(t)/A(t + T)$ is nonincreasing on $0 < t < \infty$;

(A3) $B(t) \geqslant 0$ and $B \in BC[0, \infty)$.

In the analysis which follows it is convenient to rewrite the resolvent equation (2.1) in the equivalent form

$$r(t + s, s) = a(t + s, s) - \int_0^t a(t + s, u + s)\, r(u + s, s)\, du \tag{2.11}$$

in which s occurs only as a parameter. Substituting (2.10) in (2.11) yields

$$r(t + s, s) = A(t)\, B(s) - \int_0^t A(t - u)\, B(u + s)\, r(u + s, s)\, du.$$

Thus $r(t + s, s) = y(t; s)\, B(s)$ where y is the solution of the scalar equation

$$y(t; s) = A(t) - \int_0^t A(t - u)\, B(u + s)\, y(u; s)\, du. \tag{2.12}$$

By Theorem 2 of [11] it follows that for each $s \geqslant 0$ the linear system (2.12) has a unique solution $y(t; s)$ which is locally integrable with respect to t on $0 < t < \infty$. Moreover, $y(t; s)$ is continuous in (t, s) and $0 \leqslant y(t; s) \leqslant A(t)$ for $0 < t,\ s < \infty$.

We prove the following result which uses and completes ideas of Friedman [14]. We also remark that Levin [9] has considered another example of a kernel of nonconvolution type in a different context.

THEOREM 6. *Let* (A1)-(A3) *be satisfied and let* $a(t, s)$ *be defined by* (2.10). *Then*

(i) $r(t, s)$ *satisfies* (2.4) *and* (2.6).

(ii) *if* $A(t) \to 0$ *as* $t \to \infty$, *then* $r(t, s)$ *satisfies* (2.7),

(iii) *if* $B(t)$ *is* ω*-periodic, then* $r(t, s)$ *satisfies* (2.8),

(iv) *if* $A(t) \in L^1(0, \infty)$ *or if* $\liminf_{t\to\infty} B(t) > 0$ *then* $r(t, s)$ *satisfies* (2.5), *and*

(v) *if* $A(t) \in L^1(0, \infty)$, *then* $r(t, s)$ *satisfies* (2.9).

Thus Theorem 6, together with the appropriate assumptions concerning $f(t)$ and $g(t, x)$ may be used to illustrate each of the Theorems 1, 2, 2′, 3, 4.

3. Proof of Theorem 1

Fix $\alpha > 0$ such that $\alpha B < 1$. By (H3) choose $\eta > 0$ such that

$$| g(t, x) - g(t, y) | \leqslant \alpha | x - y | \qquad (| x | , | y | \leqslant \eta), \tag{3.1}$$

uniformly in t. Fix a number $\beta > 0$ such that $\beta B \leqslant 1 - \lambda$. Using (H2) and (H3) pick $\delta > 0$ so that

$$| g(t, x) | \leqslant \beta | x | \qquad (| x | \leqslant \delta), \tag{3.2}$$

uniformly in t. Define $\mathscr{E}_0 = \min(\delta, \eta)$.

For any $\mathscr{E}$, $0 < \mathscr{E} \leqslant \mathscr{E}_0$ define

$$S_1(\mathscr{E}) = \{\varphi \in L^\infty[0, \infty) : \| \varphi \|_\infty \leqslant \mathscr{E}\}$$

and define the operator K by the relation

$$(K\varphi)(t) = y(t) - \int_0^t r(t, s)\, g(s, \varphi(s))\, ds \tag{3.3}$$

for $\varphi \in S_1(\mathscr{E})$. Using (2.4), (3.2), (3.3) and y as in the statement of Theorem 1 one obtains, for almost all t,

$$| (K\varphi)(t) | \leqslant | y(t) | + \int_0^t | r(t, s) | \beta | \varphi(s) | \, ds,$$

and

$$\| K\varphi \|_\infty \leqslant \lambda\mathscr{E} + \beta B\mathscr{E} \leqslant \lambda\mathscr{E} + (1 - \lambda)\mathscr{E} = \mathscr{E}.$$

Hence K maps $S_1(\mathscr{E})$ into itself. On the other hand using (2.4) (3.1) and (3.3) there results, for almost all t,

$$| K(\varphi_1)(t) - (K\varphi_2)(t) | \leqslant \int_0^t | r(t, s) | \alpha | \varphi_1(s) - \varphi_2(s) | \, ds,$$

and

$$\| K\varphi_1 - K\varphi_2 \|_\infty \leqslant \alpha B \| \varphi_1 - \varphi_2 \|_\infty$$

for any $\varphi_1, \varphi_2 \in S_1(\mathscr{E})$. Since $\alpha B < 1$, K is a contraction on $S_1(\mathscr{E})$. Therefore, the system (2.2), and hence also the system (1.1), has a unique solution $x \in L^\infty[0, \infty)$ with $\| x \|_\infty \leqslant \mathscr{E}$. This completes the proof of Theorem 1.

4. Proof of Theorems 2 and 2′

Fix $\alpha > 0$ such that $\alpha A < 1$ and $\alpha B < 1$. Choose $\eta > 0$ so that (3.1) is satisfied and $\beta > 0$ so that $\beta B \leqslant (1 - \lambda)$. Choose $\delta > 0$ so that (3.2) is satisfied and define $\mathscr{E}_0 = \min(\delta, \eta)$. For $0 < \mathscr{E} \leqslant \mathscr{E}_0$ define

$$S_2(\mathscr{E}) = S_1(\mathscr{E}) \cap L^1(0, \infty),$$

and define the operator K on $S_2(\mathscr{E})$ by (3.3). By the argument of the proof of Theorem 1 together with Fubini's theorem one obtains

$$\| K\varphi \|_1 \leqslant \| y \|_1 + \beta A \| \varphi \|_1 < \infty, \tag{4.1}$$

as well as

$$\| K\varphi \|_\infty \leqslant \| y \|_\infty + \beta B \| \varphi \|_\infty \leqslant \lambda\mathscr{E} + (1 - \lambda)\mathscr{E} = \mathscr{E},$$

for $\varphi \in S_2(\mathscr{E})$. Thus $K(S_2(\mathscr{E})) \subseteq S_2(\mathscr{E})$. On $S_2(\mathscr{E})$ introduce a new norm by the relation $\| \varphi \|_* = \| \varphi \|_\infty + \| \varphi \|_1$ for any $\varphi \in S_2(\mathscr{E})$. Then from (3.3) one readily obtains

$$\begin{aligned} \| K\varphi_1 - K\varphi_2 \|_* &\leqslant \alpha B \| \varphi_1 - \varphi_2 \|_\infty + \alpha A \| \varphi_1 - \varphi_2 \|_1 \\ &\leqslant \max\{\alpha A, \alpha B\} \| \varphi_1 - \varphi_2 \|_* . \end{aligned} \tag{4.2}$$

Thus K is a contraction on $S_2(\mathscr{E})$ and the proof of Theorem 2 is complete.

The proof of Theorem 2′ differs from that of Theorem 2 only in that one uses the well known inequality

$$\| a * b \|_p \leqslant \| a \|_1 \| b \|_p \qquad (1 \leqslant p < \infty),$$

where

$$(a * b)(t) = \int_0^t a(t - s)\, b(s)\, ds \qquad (0 \leqslant t < \infty),$$

$a \in L^1(0, \infty)$ and $b \in L^p(0, \infty)$. Using this inequality one readily obtains the estimates corresponding to (4.1) and (4.2).

5. Proof of Theorem 3 and Corollary 3.1

The proof of Theorem 3 is essentially the same as that of Theorem 1. One defines the set

$$S(\mathscr{E}) = \{\varphi : \varphi \in BC[0, \infty), \| \varphi \| \leqslant \mathscr{E}\},$$

and uses (2.6) to show that the operator K defined on $S(\mathscr{E})$ by (3.3) maps continuous functions into continuous functions. The remainder of the proof is unchanged when one replaces $\| \|_\infty$ by $\| \|$.

To prove Corollary 3.1 define the set $S_0 = \{\varphi : \varphi \in S(\mathscr{E}), \lim_{t\to\infty} \varphi(t) = 0\}$. Clearly S_0 is a closed subspace of S under the norm $\| \|$. We need only prove that $(K\varphi)(t) \to 0$ as $t \to \infty$ for any $\varphi \in S_0$. Let $\gamma > 0$ be given. By hypotheses choose $T > 0$ such that $| y(t) | \leqslant \gamma/3$ and $| \varphi(t) | \leqslant \gamma/(3\beta B)$ for all $t \geqslant T$. Choose $T_1 \geqslant T$ so large that by (2.7)

$$\int_0^T | r(t, s) | \, ds \leqslant \frac{\gamma}{3\beta} \qquad (t \geqslant T_1).$$

Then for $t \geqslant T_1$ one has from (2.4), (2.7), (3.2), and (3.3)

$$\begin{aligned} | K\varphi(t) | &\leqslant | y(t) | + \int_0^t | r(t, s) | \, \beta \, | \varphi(s) | \, ds \\ &\leqslant \frac{\gamma}{3} + \beta\mathscr{E} \int_0^T | r(t, s) | \, ds + \int_T^t | r(t, s) | \, \beta \, | \varphi(s) | \, ds \\ &\leqslant \frac{\gamma}{3} + \frac{\gamma}{3} + \frac{\gamma}{3B} \int_0^t | r(t, s) | \, ds \leqslant \gamma. \end{aligned}$$

Since $\gamma > 0$ is arbitrary, $(K\varphi)(t) \to 0$. Thus $KS_0 \subseteq S_0$ and one completes the proof in the usual way.

6. Proof of Lemma 4.1

For any function $f \in BC[0, \infty)$ define the operator R by the relation

$$(Rf)(t) = \lim_{n\to\infty} \int_{-n\omega}^t r(t, s) f(s) \, ds \qquad (t \geqslant 0). \tag{6.1}$$

Rf is well defined since by (2.8) one has for $n > m$

$$\int_{-n\omega}^{-m\omega} | r(t, s) | \, ds = \int_{-n\omega}^{-m\omega} | r(t + n\omega, s + n\omega) | \, ds = \int_0^{(n-m)\omega} | r(t + n\omega, s) | \, ds,$$

for any $t \geqslant 0$. In view of (2.9) this integral approaches zero as $m, n \to \infty$. Observe that if f is ω-periodic, then Rf is ω-periodic, continuous by (2.6), and hence also uniformly continuous on $0 \leqslant t < \infty$.

Define $Y(t) = f(t) - (Rf)(t)$. Clearly Y is ω-periodic. Let $\mathscr{E} > 0$ be given by. Choose $N > 0$ such that

$$\left| (Rf)(t) - \int_{-n\omega}^t r(t, s) f(s) \, ds \right| < \frac{\mathscr{E}}{2} \qquad (n \geqslant N)$$

uniformly for $0 \leqslant t < \infty$. By (2.7) choose $T > 0$ so large that

$$\| f \| \int_0^{n\omega} | r(t + n\omega, s) | \, ds < \frac{\mathscr{E}}{2} \qquad (t \geqslant T, n \geqslant N \text{ and fixed}). \tag{6.2}$$

But from (2.3)

$$\begin{aligned} y(t) - Y(t) &= \lim_{m\to\infty} \int_{-m\omega}^{0} r(t, s) f(s) \, ds \\ &= \lim_{m\to\infty} \int_0^{m\omega} r(t + m\omega, s) f(s) \, ds. \end{aligned} \tag{6.3}$$

Therefore from (6.2) and (6.3) one obtains

$$| y(t) - Y(t) | \leqslant \frac{\mathscr{E}}{2} + \int_0^{n\omega} | r(t + n\omega, s) | \, | f(s) | \, ds \leqslant \mathscr{E}$$

for t sufficiently large. Since $\mathscr{E} > 0$ is arbitrary, $y(t) - Y(t) \to 0$ as $t \to \infty$. This completes the proof of Lemma 4.1.

7. Proof of Theorem 4

The existence of a continuous solution has already been established in Theorem 3. It remains to show that the solution is asymptotically ω-periodic. Let $S(\mathscr{E})$ be the set of functions in $BC[0, \infty)$ introduced in Section 5. Let

$$S_\omega = \{\varphi \in S(\mathscr{E}) : \varphi \text{ is asymptotically } \omega\text{-periodic}\}.$$

We claim that S_ω is a closed subset of $S(\mathscr{E})$ under the norm $\| \ \|$. Suppose that $\{\varphi_n\} \subseteq S_\omega$ and $\| \varphi_n - \varphi \| \to 0$ as $n \to \infty$ for some $\varphi \in S(\mathscr{E})$. We wish to show that $\varphi \in S_\omega$. Let $\varphi_n \sim P_n$ $(n = 1, 2, 3, \ldots)$ where P_n is ω-periodic. We first show that $\{P_n\}$ is a Cauchy sequence.

For any $\eta > 0$ let N be an integer chosen so that

$$\| \varphi_n - \varphi_m \| \leqslant \frac{\eta}{3} \qquad (m, n \geqslant N).$$

By definition for each pair $(n, m) \geqslant N$ there exists a number $T = T(n, m)$ such that

$$| \varphi_n(t) - P_n(t) | \leqslant \frac{\eta}{3}, \qquad | \varphi_m(t) - P_m(t) | \leqslant \frac{\eta}{3} \qquad (t \geqslant T).$$

Thus for $t \geqslant T$,

$$\begin{aligned} | P_n(t) - P_m(t) | \leqslant | P_n(t) - \varphi_n(t) | &+ | \varphi_n(t) - \varphi_m(t) | \\ &+ | \varphi_m(t) - P_m(t) | \leqslant \eta. \end{aligned}$$

Since each P_n is ω-periodic one has $\| P_n - P_m \| \leqslant \eta$ for all $n, m \geqslant N$. Thus $\{P_n\}$ is a Cauchy sequence in $BC[0, \infty)$.

Let $P(t) = \lim P_n(t)$. Clearly P is ω-periodic. We will show that $\varphi \sim P$. Let $\eta > 0$ be given. There is an integer n such that $\| \varphi - \varphi_n \| \leqslant \eta/3$ and $\| P - P_n \| \leqslant \eta/3$. For such an integer n there is a $T = T(n) > 0$ such that $| \varphi_n(t) - P_n(t) | \leqslant \eta/3$ for $t \geqslant T$. Thus for $t \geqslant T$ one has

$$| \varphi(t) - P(t) | \leqslant \| \varphi - \varphi_n \| + | \varphi_n(t) - P_n(t) | + \| P_n - P \| \leqslant \eta.$$

Thus $\varphi \in S_\omega$ and S_ω is a closed subset of $S(\mathscr{E})$.

By hypothesis $y \sim Y$ for some ω-periodic function Y. If $\varphi \in S_\omega$, then $\varphi \sim \Phi$ where Φ is some ω-periodic function. If the operator K is defined by (3.3) for all $\varphi \in S_\omega$, then we claim that $K\varphi \sim X$ where

$$X(t) = Y(t) + \lim_{n\to\infty} \int_{-n\omega}^{t} r(t, s)\, g(s, \Phi(s))\, ds. \tag{7.1}$$

Since $\varphi \in S_\omega$ and g is continuous in x and ω-periodic in t, the limit in (7.1) is well defined (see Definition (6.1)). Moreover, X is ω-periodic by the argument of Lemma 4.1. To see that $K\varphi \sim X$ let $\eta > 0$ be given. Choose an integer $M > 0$ such that $| y(t) - Y(t) | < \eta$ and $| \varphi(t) - \Phi(t) | < \eta$ for all $t \geqslant M\omega$. Now choose an integer $N > M$ such that

$$\left| \lim_{n\to\infty} \int_{-n\omega}^{t} r(t, s)\, g(s, \Phi(s))\, ds - \int_{-N\omega}^{t} r(t, s)\, g(s, \Phi(s))\, ds \right| < \eta.$$

Select a number $T > N\omega$ such that

$$\int_0^{N\omega} | r(t + N\omega, s) |\, ds < \eta, \qquad \int_0^{M\omega} | r(t, s) |\, ds < \eta \qquad (t \geqslant T).$$

Then for $t \geqslant T$ one obtains successively

$$\begin{aligned}
|K\varphi(t) - X(t)| &\leqslant | y(t) - Y(t) | + \eta + \int_{-N\omega}^{0} | r(t, s)\, g(s, \Phi(s)) |\, ds \\
&\qquad + \int_0^t | r(t, s) |\, | g(s, \varphi(s)) - g(s, \Phi(s)) |\, ds \\
&\leqslant 2\eta + \int_0^{N\omega} | r(t + N\omega, s) |\, | g(s, \Phi(s)) |\, ds \\
&\qquad + \int_0^t | r(t, s) |\, \alpha\, | \varphi(s) - \Phi(s) |\, ds \\
&\leqslant 2\eta + \beta\mathscr{E} \int_0^{N\omega} | r(t + N\omega, s) |\, ds \\
&\qquad + (2\alpha\mathscr{E}) \int_0^{M\omega} | r(t, s) |\, ds + \alpha\eta \int_{M\omega}^{t} | r(t, s) |\, ds \\
&\leqslant (2 + \beta\mathscr{E} + 2\alpha + B\alpha)\, \eta.
\end{aligned}$$

Since $\eta > 0$ is arbitrary and α, β, $\mathscr{E}$ and B are preassigned constants, the last inequality shows that $K\varphi(t) - X(t) \to 0$ as $t \to \infty$. Thus $KS_\omega \subseteq S_\omega$ and this completes the proof in the familiar way.

8. Proof of Corollary 4.2

It follows from Theorem 4 that the solution $x(t)$ exists and is asymptotically constant. Moreover from the proof of Theorem 4 one has for any $\omega > 0$

$$\begin{aligned} x(\infty) &= y(\infty) + \lim_{n\to\infty} \int_{-n\omega}^{t} r(t-s)\, g(x(\infty))\, ds \\ &= y(\infty) + \left(\int_0^\infty r(s)\, ds\right) g(x(\infty)), \end{aligned}$$

as asserted.

9. Proof of Theorem 5

By Theorem 3 one has again that the solution $x(t)$ of (1.1) exists on $0 \leqslant t < \infty$ and $\| x \| \leqslant \mathscr{E}$. By an argument paralleling the proof of Lemma 4.1 in Section 6 one shows that for f almost periodic the solution $y(t)$ of the linear system (1.2) given by (2.3) is asymptotically almost periodic, $\mathrm{Mod}_n(f)$. Indeed $y \approx Y$ where

$$Y(t) = f(t) - \int_{-\infty}^{t} r(t-s) f(s)\, ds.$$

Define

$$S_a = \{\varphi \in S(\mathscr{E});\ \varphi \text{ is asymptotically almost periodic } \mathrm{Mod}_n(f, g)\},$$

where $S(\mathscr{E})$ is defined in Section 5. As in the proof of Theorem 4 in Section 7 one shows that S_a is a closed subset of $S(\mathscr{E})$. Moreover, if $\varphi \in S_a$ and $\varphi \approx \Phi$, then $K\varphi \approx X$ where

$$X(t) = Y(t) - \int_{-\infty}^{t} r(t-s)\, g(s, \Phi(s))\, ds.$$

Thus $KS_a \subseteq S_a$ and this completes the proof.

10. Proof of Theorem 6

Let $\beta = \| B \|$ and consider the scalar equation for R,

$$R(t) = \beta A(t) - \beta \int_0^t A(t-u)\, R(u)\, du. \tag{10.1}$$

LEMMA 6.1. *If either* $A \in L^1[0, \infty)$ *or* $\liminf B(t) > 0$ $(t \to \infty)$ *then* $r(t + s, s) \in L^1[0, \infty)$ *as a function of* t *and*

$$\sup \left\{ \int_0^\infty | r(t + s, s) | \, dt : s \geqslant 0 \right\} < \infty.$$

PROOF. Since $r(t + s, s) = y(t; s) B(s)$ and $B \in BC[0, \infty)$, it suffices to show that as a function of t, $y \in L^1[0, \infty)$ and satisfies the desired bound. By Theorem 2 of [11] the solution $R(t)$ of (10.1) exists and is continuous on $0 < t < \infty$. Moreover, $0 \leqslant R(t) \leqslant \beta A(t)$, $\int_0^\infty R(t)\, dt \leqslant 1$ and the integral of R equals one if and only if $A \notin L^1(0, \infty)$.

For any $\varphi \in L^1[0, \infty)$ define an operator K by the relation

$$(K\varphi)(t; s) = \int_0^t R(t - u) \left(1 - \frac{B(u + s)}{\beta}\right) \varphi(u)\, du$$

for $t, s \geqslant 0$. It is easily verified that K maps $L^1[0, \infty)$ into itself and that

$$\| K \| \leqslant \| R \|_1 \left\| 1 - \frac{B}{\beta} \right\|.$$

If $A \in L^1[0, \infty)$, then $\| K \| \leqslant \| R \|_1 < 1$. If $\liminf B(t) > 0$ $(t \to \infty)$, then $\| K \| \leqslant \| 1 - B/\beta \| < 1$. In either case the map K^* defined by

$$K^*\varphi = R + K\varphi \qquad (\varphi \in L^1[0, \infty))$$

is a contraction. Thus the equation

$$Y(t; s) = R(t) + \int_0^t R(t - u) \left(1 - \frac{B(u + s)}{\beta}\right) Y(u; s)\, du \qquad (s \geqslant 0) \quad (10.2)$$

has a unique solution which is of class $L^1[0, \infty)$ with respect to t. It is elementary to verify that $y(t; s) = Y(t; s)/\beta$ is the unique solution of (2.12). From (10.2) one readily has

$$0 \leqslant \int_0^\infty y(t; s)\, dt \leqslant \beta \| R \|_1 (1 - \| K \|)^{-1}. \quad (10.3)$$

Since the bound (10.3) is independent of $s \geqslant 0$, this completes the proof of Lemma 6.1.

To prove the first conclusion of Theorem 6 consider the integral equation

$$Z(t) = 1 - \int_0^t A(t - s) B(s) Z(s)\, ds \qquad (t \geqslant 0). \quad (10.4)$$

If $r(t, s) = y(t; s) B(s)$ is the resolvent kernel corresponding to the kernel $A(t - s) B(s)$, then the solution of (10.4) is given by

$$Z(t) = 1 - \int_0^t r(t, s)\, ds, \tag{10.5}$$

(see (2.3) with $f(t) \equiv 1$). By Theorem 1 of [11] the solution $Z(t)$ is non-negative. Since $r(t, s)$ is also nonnegative (2.4) is satisfied with the constant $B = 1$,

$$0 \leqslant \int_0^t | r(t, s) |\, ds = \int_0^t r(t, s)\, ds \leqslant 1 \qquad (0 < t < \infty).$$

To show that (2.6) is satisfied fix $t \geqslant 0$ and h such that $t + h \geqslant 0$. Since $| r(t, s) | \leqslant \beta A(t - s)$ it is clear from (A1) that

$$\lim_{h \to 0} \int_t^{t+h} | r(t + h, s) |\, ds = 0.$$

From (2.1) one has

$$\begin{aligned} \int_0^t | r(t + h, s) - r(t, s) |\, ds \leqslant & \int_0^t | A(t + h - s) - A(t - s) | B(s)\, ds \\ & + \int_0^t \int_s^t | A(t + h - u) - A(t - u) | B(u) r(u, s)\, du\, ds \\ & + \int_0^t \int_t^{t+h} A(t + h - u) B(u) r(u, s)\, du\, ds = I_1 + I_2 + I_3 . \end{aligned}$$

For a fixed $t \geqslant 0$ one has

$$\begin{aligned} | I_1 | & \leqslant \beta \int_0^t | A(s + h) - A(s) |\, ds \to 0 \qquad (h \to 0), \\ | I_2 | & \leqslant \beta^2 \int_0^t \int_s^t | A(t + h - u) - A(t - u) | A(u - s)\, du\, ds \\ & \leqslant \beta^2 \left(\int_0^t A(s)\, ds \right) \int_0^t | A(s + h) - A(s) |\, ds \to 0, \end{aligned}$$

and

$$\begin{aligned} | I_3 | & \leqslant \beta^2 \int_0^t \int_t^{t+h} A(t + h - u) A(u - s)\, du\, ds \\ & \leqslant \beta^2 \left(\int_0^{t+h} A(s)\, ds \right) \int_0^h A(h - s)\, ds \to 0. \end{aligned}$$

Thus (2.6) is satisfied.

That (ii)-(v) are satisfied follows easily from the fact that $0 \leqslant r(t, s) \leqslant A(t-s)\, B(s)$ and, in the case of (2.8), from (2.1) and that $B(t)$ is ω-periodic. Part (ii) is included in Lemma 6.1. This completes the proof of Theorem 6.

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